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On Congruence Lattices of Regular Semigroups

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The θ relation, on the congruence lattice Λ of a regular semigroup S, identifies congruences whose restrictions to the set E of idempotents of S are the same. It is a now classical result that θ is a complete congruence on Λ. In this paper we consider a decomposition of Λ into a product of Λ/θ and the lattices of normal subgroups of the maximal subgroups of S, and use this to discuss semimodularity, modularity and distributivity of Λ. When S is finite or, more generally, when S satisfies $\min_E$, the minimal condition on idempotents, or is completely regular, Λ can always be "co-ordinatized" by such a decomposition; we characterize (Theorem 2.7), in terms of a very readily testable condition on Green's relations, those S for which the decomposition is *subdirect*, or, equivalently, for which each map which takes congruences to their restrictions on a (fixed) maximal subgroup is a morphism. The condition in question, which we call (A), specializes in inverse (in fact in pseudo-inverse) semigroups, for example, to (A'): *for any* e, f *in* E, $f < e$, *such that* $eSe \cap J_f$ *is a group*, $fa = f$ *for all* $a \mathscr{H} e$. (Various reformulations are given in Section 2).

Now if Λ is semimodular then the decomposition is *necessarily* subdirect (Theorem 2.8). From this we obtain the following: *Λ is semimodular if and only if Λ/θ is semimodular and S satisfies* (A). Similar characterizations are obtained for modularity and distributivity.

For *inverse* semigroups a very satisfactory result is obtained by means of Theorem 3.3: Λ/θ is *always* semimodular when S satisfies $\min_E$ or, more generally, when S is hypersemisimple (that is, each morphic image of S is completely semisimple). Thus *an inverse semigroup satisfying* $\min_E$ *has semimodular congruence lattice if and only if S satisfies* (A'). For example, the congruence lattice of any finite combinatorial inverse semigroup is semimodular. Under similar hypotheses, we also characterize modularity and distributivity of Λ/θ (Theorem 3.5): these are equivalent, and equivalent to (B): *for any* e, f *in* E, $f < e$, *such that* $eSe \cap J_f$ *is a group*, f *is comparable with any idempotent of* S *less than* e. (This generalizes known charac-

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terizations [2, 18] of *semilattices* whose congruence lattices are modular (equivalently, distributive).) Thus Λ is modular, for example, if and only if S satisfies both (A') and (B) (Corollary 3.7).

In Section 4, a decomposition of Λ/θ itself, somewhat analogous to that of Λ, is discussed for arbitrary regular semigroups. A "co-ordinatization" can again be obtained. For pseudo-inverse semigroups (regular semigroups S in which eSe is inverse for each $e \in E$) we prove analogues of the theorems in Section 2 and extensions of those in Section 3. The decomposition of Λ/θ is subdirect if and only if S satisfies a condition (C) similar in style to (A') (Theorem 5.3) and it follows, for example, that Λ is semimodular if and only if S satisfies both (A') and (C). The problems involved in extending these results to regular semigroups in general are discussed in Section 4.

In the final section we show how results of Spitznagel [23] on "θ-modular" bands of groups can be quickly deduced from the results in Section 2.

It is interesting to observe that Zitomirskiĭ [24] has obtained superficially similar results on the modularity of congruence lattices of inverse semigroups. It appears that, in fact, there is little overlap between our results; however the possibility of common generalization is suggested. (The author thanks Simon Goberstein for this reference).

1. Preliminaries

We collect the definitions and results which will be required in the sequel. If S is a semigroup we denote its set of idempotents by E (and denote the set of idempotents of any subset A of S by $E(A)$). It is well known (see [12, Chapter 1]) that the set $\Lambda(S)$, (or just Λ) of congruences on S forms a complete lattice under join $\vee$ and meet $\cap$, with least element ι and greatest element ω. If $A \subseteq S$ and $\rho \in \Lambda$ we denote by $\rho|A$ the restriction of ρ to A.

The relation θ on a regular semigroup S is defined by $\rho\theta\tau$ if $\rho|E = \tau|E$. This relation was introduced by Reilly and Scheiblich, who proved the major part of the following important theorem.

Result 1.1 [20, 21, 7]. Let S be a regular semigroup. Then θ is a complete lattice congruence on Λ. ∎

The following result is well known. In the sequel we will show how, under certain finiteness hypotheses, it may be substantially improved.

Result 1.2. Let S be a regular semigroup and $\rho, \tau \in \Lambda$. Then $\rho \subseteq \tau$ if and only if $e\rho \subseteq e\tau$ for all $e \in E$. ∎

The following proposition is easily proved. The surjectivity of the map in question is the essence of Proposition 4.5 of [10].

PROPOSITION 1.3. *Let S be an arbitrary semigroup and suppose $e \in E$. The map $\rho \mapsto \rho \mid eSe$ is a complete lattice morphism of Λ upon $\Lambda(eSe)$.* ▮

If I is an ideal of a semigroup S we denote by ρ_I the Rees congruence on S modulo I.

RESULT 1.4 [15, Propositions 10.1, 10.2]. Let S be an arbitrary semigroup and I an ideal of S. For any $\rho \in \Lambda$, and $x, y \in S$, $(x, y) \in \rho \vee \rho_I$ if and only if $x\rho a$ and $y\rho b$ for some $a, b \in I$. If S is regular then ρ_I "distributes over join," that is, $\rho_I \cap (\rho \vee \tau) = (\rho_I \cap \rho) \vee (\rho_I \cap \tau)$ for all $\rho, \tau \in \Lambda$. ▮

The following important property of congruences on regular semigroups is known as Lallement's lemma. (Throughout, $V(x)$ denotes the set of inverses of an element x in a regular semigroup.)

RESULT 1.5. Let S be a regular semigroup and $\rho \in \Lambda$. If $a \in S$ and $a\rho$ is an idempotent of S/ρ then for any $x \in V(a^2)$, $a\rho axa$, an idempotent of S.

COROLLARY 1.6. *Let S be a regular semigroup and suppose $e\rho x$ for some $\rho \in \Lambda$, $e \in E$ and $x \in S$ such that $J_x < J_e$. Then $e\rho f$ for some $f \in E$, $J_f < J_e$.*

We now go on to consider some of the finiteness conditions which appear as hypotheses in the sequel. (For elementary definitions and properties of semigroups in general, and regular semigroups in particular, see [12]).

We call a semigroup S (necessarily regular) *hypersemisimple* if (S and) all of its morphic images are completely semisimple. (A semigroup is *completely semisimple* if each of its principal factors is completely (0-) simple). The crucial property of hypersemisimple semigroups we require is

LEMMA 1.7. *Let S be a hypersemisimple semigroup and $e, f \in E$, $e > f$. If $e\rho f$ for some $\rho \in \Lambda$ then $e\rho g$ for every $g \in E$, $g\mathscr{J}f$ such that $g < e$.*

Proof. Let $g \in E$, $g\mathscr{J}f$, $g < e$. Then $g\rho\mathscr{J}f\rho = e\rho$ and $g\rho \leqslant e\rho$ in S/ρ. From complete semisimplicity of S/ρ we obtain $g\rho = e\rho$. ▮

(T. E. Hall has shown (unpublished) that a regular semigroup is hypersemisimple *if and only if* it has the property described in the lemma, but we will not use this fact).

Following [11] a semigroup S is called *group-bound* if some power of each element of S belongs to a subgroup of S. Of course every *periodic*, and thus every *finite* semigroup is group bound. So is every *completely regular* semigroup (union of groups). Since it is clear that a group-bound semigroup cannot contain a copy of the bicyclic semigroup, so that every regular principal factor is completely (0-) simple, a regular group-bound semigroup is completely semisimple. Moreover a morphic image of a group bound semigroup is again group bound: thus *a regular group bound semigroup is hypersemisimple.*

A semigroup is said to satisfy $\min_E$ if its set of idempotents satisfies the minimal condition (under the natural partial order defined by $e \leqslant f$ if $ef = fe = e$). From [11] *a regular semigroup satisfying* $\min_E$ *is group-bound.*

Our primary interest, although we will rarely make explicit mention of it, is in finite regular and in completely regular semigroups; we prove our results under various combinations of the above hypotheses, however.

Finally we survey briefly the purely lattice theoretic results needed. For elementary definitions and results the reader is referred to [6]. In particular, if a and b are elements of a lattice L with $a < b$, we denote by $[a, b]$ the *interval* (sublattice) $\{x \in L\colon a \leqslant x \leqslant b\}$ of L and say that b *covers* a, $b \succ a$, if $[a, b] = \{a, b\}$.

A lattice L is called *M-symmetric* if the modularity relation M, defined by

$$aMb \quad \text{if} \qquad x \vee (a \wedge b) = (x \vee a) \wedge b \qquad \text{for all} \quad x \leqslant b,$$

is symmetric. (Thus, every modular lattice (in which, by definition, aMb for all a and b) is M-symmetric). The lattice L is *semimodular* if $a \succ a \wedge b$ implies $a \vee b \succ b$ for all a, b. Every M-symmetric lattice is semimodular and although in general the converse is not true the author has shown [16] that in any algebraic lattice (and thus in any congruence lattice) with the descending chain condition (D.C.C.), the two concepts coincide. One consequence of our results will be that, under the hypotheses of this paper, the two concepts also coincide in the congruence lattices of pseudoinverse semigroups.

Another form of "semimodularity" we will call here the *double covering property*: $a \succ a \wedge b$ and $b \succ a \wedge b$ together imply $a \vee b \succ a$ and $a \vee b \succ b$. Again, in algebraic lattices with D.C.C. this property coincides with those above. In [15] the author shows that, even in the congruence lattices of *inverse* semigroups, (not, however, satisfying the hypotheses of this paper), double covering and semimodularity need not coincide. Our main interest will be in M-symmetry and semimodularity (and in modularity and distributivity); we shall therefore make only occasional mention of analogues, for double covering, of our results.

That M-symmetry and semimodularity are preserved by interval sublattices, but not by sublattices in general, is well-known. We now prove some further preservation properties, of independent interest.

PROPOSITION 1.8. *M-symmetry and semimodularity are preserved by subdirect products.*

Proof. That M-symmetry is preserved by subdirect products is the content of [13, Proposition 5.1]. Let L_i be a family of semimodular lattices and suppose L is a subdirect product of the family. Thus, there are morphisms ϕ_i of L upon L_i such that $a\phi_i = b\phi_i$ for all i implies $a = b$. Suppose $a, b \in L$ with $a \succ a \wedge b$. We may assume $a \not\geqslant b$. Let $i \in I$ andput $x\phi_i = x_i$ for all $x \in L$. It is easily seen that either $a_i = a_i \wedge b_i$ or $a_i \succ a_i \wedge b_i$. In the latter case we have $a_i \vee b_i \succ b_i$, by semimodularity of L_i. Let $z \in [b, a \vee b]$, $z \neq a \vee b$. Then $z_i \in [b_i, a_i \vee b_i]$. If $a_i = a_i \wedge b_i$ then $z_i = b_i$; otherwise either $z_i = b_i$ or $z_i = a_i \vee b_i \geqslant a_i$. Now since $a \succ a \wedge b$ and $z \not\geqslant a$, $a \wedge z = a \wedge b$, whence if $z_i \geqslant a_i$ we have $a_i = a_i \wedge z_i = a_i \wedge b_i$ and $z_i = b_i$ once more. Hence $z_i = b_i$ for all i, that is, $z = b$. Therefore $a \vee b \succ b$ and L is semimodular. ∎

PROPOSITION 1.9. *Let L be a complete semimodular lattice and ϕ a complete lattice morphism of L upon a lattice M. Then M is semimodular.*

Proof. Suppose first that $x, y \in M$ with $x \succ y$. We may choose $a, b \in L$, $a > b$, such that $a\phi = x$ and $b\phi = y$. Let $n = \bigvee \{c \in L : b \leqslant c < a \text{ and } c\phi = y\}$ and $m = \bigwedge \{d \in L : n < d \leqslant a \text{ and } d\phi = x\}$. Then $m\phi = x$, $n\phi = y$ and $m \succ n$ in L.

Thus if $x, y \in M$ are such that $x \succ x \wedge y$ and $x \not\geqslant y$ there exist $a, b \in L$, $a \succ b$, such that $a\phi = x$ and $b\phi = x \wedge y$. Let $c\phi = y$, $c \in L$. Since $(c \vee b)\phi = y \vee (x \wedge y) = y$, we may assume $c > b$. Now $a \geqslant c \wedge a \geqslant b$ and $c \not\geqslant a$ (since $y \not\geqslant x$), so $c \wedge a = b$, that is, $a \succ c \wedge a$. By semimodularity of L, $a \vee c \succ c$. Now suppose $x \vee y \geqslant z \geqslant y$ and let $d\phi = z$, $d \in L$; since $((d \wedge (a \vee c)) \vee c)\phi = z$ we may assume $d \in [c, a \vee c]$, so that either $d = c$ or $d = a \vee c$, yielding $z = y$ or $z = x \vee y$ respectively. Thus $x \vee y \succ y$ and M is semimodular. ∎

A similar result holds for double covering but we do not know whether it is so for M-symmetry. The following application of Corollary 1.8 will be used in Section 2 to circumvent this problem.

COROLLARY 1.10. *If the lattice L is isomorphic with a subdirect product of two lattices L_1 and L_2, where L_2 is modular, then L is M-symmetric if and only if L_1 is.*

Proof. Sufficiency is clear. Now let L_1 be M-symmetric and suppose aMb in L. Denote by ϕ_1 and ϕ_2 the projections of L upon L_1 and L_2 respectively. As in the proof of Proposition 5.1 of [13], $a\phi_1 M b\phi_1$, whence by M-symmetry $b\phi_1 M a\phi_1$. But by modularity of L_2, $b\phi_2 M a\phi_2$. So (as in the proof of the same proposition) bMa. Thus L is M-symmetric. ∎

2. Decompositions of Λ

LEMMA 2.1. *Let S be a regular semigroup and $\rho, \tau \in \Lambda$. Then $\rho \subseteq \tau$ if and only if $\rho\theta \leqslant \tau\theta$ and whenever $(a, e) \in \mathscr{R} \cap \rho$, $a, e \in S$, $e \in E$, $(a, e) \in \mathscr{R} \cap \tau$.*

Proof. Necessity is clear. Conversely, suppose $a\rho e$, $a, e \in S$, $e \in E$. Then by Result 1.5 $a\rho g = axa \in E$, where $x \in V(a^2)$; since $e\rho g$ we have $e\tau g$. Further, $g\rho ga$ and $g\mathscr{R}ga$ since if $a' \in V(a)$, $aa'g = g$; thus $g\tau ga$. But $a'a\rho a'ga$, with both $a'a$ and $a'ga$ in E, so that $a'a\tau a'ga$ and $a\tau ga$. Hence $e\tau a$. The proof is completed by an application of Result 1.2. ∎

A lemma in a similar vein will be proved in Section 4.

COROLLARY 2.2. *Let S be a regular group-bound semigroup and $\rho, \tau \in \Lambda$. Then $\rho = \tau$ if and only if $\rho\theta\tau$ and $\rho | H_e = \tau | H_e$ for all $e \in E$.*

Proof. Let $(a, e) \in \rho \cap \mathscr{R}$, $a \in S$, $e \in E$. For some $n \geqslant 1$ we have $a^n \mathscr{H} f$ for some $f \in E$ and since $a^{2n}\rho a^n \rho a\rho e$, it follows that $f\rho e$, so $f\tau e$ also. But $a = ea\tau a^{2n-1}a = a^{2n}\tau f$ (since $(a^{2n}, f) \in \rho \mid H_f$) and thus $a\tau e$. The result now follows from the lemma. ∎

When S is group-bound there is, therefore, a one-one map

$$\phi : \rho \mapsto (\rho\theta, \{\rho | H_e : e \in E\})$$

of Λ into the direct product of Λ/θ and $\{\Lambda(H_e) : e \in E\}$. Since, by Result 1.1, the component map $\rho \mapsto \rho\theta$ is a (complete) surjective morphism, and, as we later observe, each $\rho \mapsto \rho | H_e$ is also surjective in the most interesting cases, ϕ will be a morphism or, equivalently, represent Λ as a subdirect product, if and only if each $\rho \to \rho | H_e$ is a morphism.

In completely semisimple semigroups one characterization of this property is as follows.

PROPOSITION 2.3. *Let S be a completely semisimple semigroup and $e \in E$. The map $\rho \mapsto \rho | H_e$ is a morphism if and only if $H_e \subseteq e\rho$ for all $\rho \in \Lambda$ such that $e\rho f$ for some $f < e$.*

Proof. Suppose $\rho \mapsto \rho | H_e$ is a morphism, $e\rho f$, $f < e$ and $a\mathscr{H}e$. Let I denote the ideal $\{x \in S : J_x < J_e\}$ of S. By complete semisimplicity f and fa belong to I, and since $e\rho f$, $a\rho fa$. Thus $(e, a) \in (\rho \vee \rho_I) | H_e = \rho | H_e \vee \rho_I | H_e = \rho | H_e$, whence $a\rho e$. Therefore $H_e \subseteq e\rho$.

Conversely, suppose $\rho, \tau \in \Lambda$. Clearly $(\rho \cap \tau) | H_e = \rho | H_e \cap \tau | H_e$ and

$(\rho \vee \tau)|H_e \supseteq \rho|H_e \vee \tau|H_e$. To show the remaining inclusion let $(a, e) \in (\rho \vee \tau)|H_e$. Suppose $(a, e) \notin \rho \cup \tau$. In that case there is a sequence

$$a = x_0 \to x_1 \to \cdots \to x_n = e,$$

where each $x_i \in eSe$, without loss of generality, and each $(x_{i-1}, x_i) \in \rho \cup \tau$. Suppose some first $x_i \notin H_e$, so $x_{i-1} \in H_e$ and $x_{i-1} \rho x_i$, say. Let $x'_{i-1} \in V(x_{i-1}) \cap H_e$; then $e = x_{i-1} x'_{i-1} \rho x_i x'_{i-1}$, with $J_{x_i x'_{i-1}} < J_e$. Applying Corollary 1.7, $e\rho f$ for some $f < e$ so that, by hypothesis, $a\rho e$, contradicting our assumption. Therefore, each $x_i \in H_e$, that is, $(a, e) \in \rho|H_e \vee \tau|H_e$. ∎

We now show that under suitable hypotheses an intrinsic characterization of the above property is provided by the following property, which we introduce first for individual idempotents.

Let $e \in E$. We say that S *satisfies* (A) *for* e if *for any* $f \in E$, $f < e$, *such that* $eSe \cap J_f$ *is completely simple*, $faf\sigma f$ *for all* $a\mathscr{H}e$ (σ denoting the least group congruence on $eSe \cap J_f$); S satisfies (A) if it satisfies (A) for all $e \in E$.

Various alternative formulations and specializations will be discussed at the end of this section. We note, however, that when $eSe \cap J_f$ is completely simple, $faf\mathscr{H}f$ for all $a\mathscr{H}e$.

LEMMA 2.4. *Let S be completely semisimple, $e \in E$, and suppose $\rho \mapsto \rho|H_e$ is a morphism. Then S satisfies (A) for e.*

Proof. Suppose $eSe \cap J_f$ is completely simple, $f \in E$, $f < e$, and let $a\mathscr{H}e$. Consider the congruence $\beta = (e, f)^*$ generated by the pair (e, f). By the previous proposition $e\beta a$, whence $f = fef\beta faf$. There is therefore a sequence

$$f = x_0 \to x_1 \to \cdots \to x_n = faf$$

with each $x_i \in fSf$, without loss of generality, and each (x_{i-1}, x_i) has the form $(f(s_i e t_i)f, f(s_i f t_i)f)$ or $(f(s_i f t_i)f, f(s_i e t_i)f)$ for some $s_i, t_i \in S^1$. Let $i \geqslant 1$ and suppose $x_{i-1}\mathscr{H}f$ and $x_{i-1}\sigma f$. If $x_{i-1} = f(s_i e t_i)f = (fs_i e)\, e(et_i f)$ then $fs_i e$ and $et_i f$ belong to $eSe \cap J_f$. By complete simplicity, $f\mathscr{H}(fs_i e) f(et_i f) = x_i$. If g and h denote the identities of the subgroups containing $fs_i e$ and $et_i f$ respectively (so g and $h \in E(eSe \cap J_f)$), then $f\sigma x_{i-1} = (fs_i e)(et_i f) = (fs_i e)\, gh(et_i f)\, \sigma\, (fs_i e) f(et_i f) = x_i$. If, on the other hand, $x_{i-1} = f(s_i f t_i) f$ a similar result is obtained. Thus by induction $f\sigma x_n = faf$, as required. ∎

To prove a converse to Lemma 2.4 an additional hypothesis (†), clearly relevant to (A), is required: we say that S satisfies (†) if *whenever* $e, f \in E$, $e > f$, *and* $e\rho f$ *for some* $\rho \in \Lambda$, *then* $e\rho g$ *for some* $g \leqslant f$ *such that* $eSe \cap J_g$ *is completely simple.*

Clearly any completely regular semigroup satisfies (†).

LEMMA 2.5. *A regular semigroup S satisfying* $\min_E$ *satisfies* (†).

Proof. Suppose $e, f \in E$, $e > f$, and $e\rho f$ for some $\rho \in \Lambda$. Choose g minimally in E so that $e\rho g$, and suppose $eSe \cap J_g$ is not completely simple, so that for some $h \in E(eSe \cap J_g)$, $J_{gh} < J_g$. Now by the results of Section 1, S is hypersemisimple and so $e\rho h$ by Lemma 1.8. Thus $g\rho gh$ and, from Corollary 1.6, $g\rho k$ for some $k \in E$, $k < g$, contradicting the minimality of g. ∎

Examples may be constructed of regular group-bound semigroups which do not satisfy (†).

LEMMA 2.6. *Let S be a hypersemisimple semigroup with* (†) *and let $e \in E$. If S satisfies (A) for e then the map $\rho \mapsto \rho | H_e$ is a morphism.*

Proof. Suppose $f \in E, f < e$ and $e\rho f$ for some $\rho \in \Lambda$. By Proposition 2.3 it is sufficient to show $H_e \subseteq e\rho$. So let $a\mathscr{H}e$. Now from (†) we obtain an idempotent $g < f$ such that $e\rho g$ and $eSe \cap J_g$ is completely simple, whence by (A) $gag\sigma g$. From Lemma 1.7, however, $e\rho h$ for all $h \in E(eSe \cap J_g)$, so that $\rho | eSe \cap J_g$ is a group congruence, and thus contains σ. So $gag\rho g\rho e$. But $a = eae \, \rho \, gag \, \rho e$, as required. ∎

To complete the proof of the following theorem summarizing, in the most interesting cases, the above results, we show that given a maximal subgroup H_e of a completely semisimple semigroup S, any congruence ρ on H_e extends to a congruence on S. We may first extend ρ to the congruence $\bar{\rho}$ on eSe whose classes are those of ρ together with the ideal $eSe \backslash H_e$ of eSe. By Proposition 1.3 $\bar{\rho}$ may be extended to a congruence on S.

We have therefore proven

THEOREM 2.7. *Let S be a regular semigroup which either has* $\min_E$ *or is completely regular. Then the following are equivalent*:

(a) *the map $\rho \mapsto (\rho\theta, \{\rho | H_e \colon e \in E\})$ is an isomorphism of Λ upon a subdirect product of the lattices Λ/θ and $\Lambda(H_e)$, $e \in E$,*

(b) *each map $\rho \mapsto \rho | H_e$ is a morphism,*

(c) *S satisfies (A).* ∎

We now use the theorem to study lattice-theoretic properties of Λ. Note first that, of course, $\Lambda(H_e)$ is isomorphic to the lattice of normal subgroups of H_e, which is modular. Thus, given (A), Λ is isomorphic with a subdirect product of Λ/θ together with a certain modular lattice. Hence if Λ/θ is M-symmetric, semimodular or modular then Λ has the same property, and if Λ/θ is distributive and each $\Lambda(H_e)$ is distributive, so is Λ.

Rather remarkably, semimodularity (the weakest of those properties) *implies* (A), leading to the following characterizations. (The reader is referred to Section 1 for lattice-theoretic definitions and properties).

THEOREM 2.8. *Let S be a regular semigroup. If S is hypersemisimple and Λ is semimodular then S satisfies (A).*

Hence if, further, S has $\min_E$ *or is completely regular, then Λ is semimodular, M-symmetric, modular (or distributive) if and only if S satisfies (A) and Λ/θ is semimodular, M-symmetric, modular (or distributive, and the lattice of normal subgroups of each maximal subgroup is distributive) respectively.*

Proof. The final statement follows from the comments preceding the theorem, from Proposition 1.9 (in conjunction with Result 1.1) and from Corollary 1.10 and Proposition 1.8.

To prove the first statement, suppose Λ is semimodular and let $e, f \in E$, $f < e$, with $eSe \cap J_f$ completely simple. Note that $\Lambda(eSe)$ is also semimodular, by Propositions 1.9 and 1.3. So is $\Lambda(eSe/K)$, where K is the ideal $\{x \in eSe\colon J_x \not\geqslant J_f\}$ of eSe, since $\Lambda(eSe)/K \cong [\rho_K, \omega]$, an interval of $\Lambda(eSe)$). Clearly S satisfies (A) for e if and only if eSe/K does, so without loss of generality we may assume S is a monoid with identity e, and with $J = J_f$ its unique 0-minimal $\mathscr{J}$-class; then J is a completely simple subsemigroup of S.

Suppose $e\rho f$ for some $\rho \in \Lambda$. By Proposition 2.3 and Lemma 2.4 it is sufficient to show $H_e \subseteq e\rho$. Now ρ is in fact a group congruence on the subsemigroup $S^* = S\backslash 0$ of S, for if $h \in E(S^*)$ then by [8, Theorem 1], $h \geqslant g$ for some $g \in E(J)$ and, using Lemma 1.7, $e\rho g$, so $e\rho h$. Denote by I the ideal $S\backslash H_e$. Then $S/(\rho \cap \rho_I)$ is a semilattice of two groups $H_{\bar{e}}$ and $H_{\bar{f}}$ (possibly ith zero adjoined), where we put $\bar{x} = x(\rho \cap \rho_I)$ for all $x \in S$. But $\Lambda(S/(\rho \cap \rho_I))$ is again semimodular (as above). By Theorem 3.5 of [4] (where "semimodularity" actually means double covering, in our terminology), the linking morphism $x \mapsto fx$, $x \in H_{\bar{e}}$, is constant. Thus for any $a\mathscr{H}e$, $\bar{f}\bar{a} = \bar{f}$, that is, $fa\rho f$. Since $e\rho f$ and $a\rho fa$, $e\rho a$ as required. ∎

It is easily seen that "semimodularity" may be replaced by "double covering" in the statement of the theorem.

We conclude the section with some reformulations and specializations of (A). First we remark that the least group congruence σ on a completely simple semigroup T is easily described. (When T is written as a Rees matrix semigroup, see [1, Section 3.4]). If f is an idempotent of T then $f\sigma \cap H_f$ is the normal subgroup of H_f generated by $\langle E \rangle \cap H_f$, where $\langle E \rangle$ is the subsemigroup of T generated by E. Thus an *orthodox* semigroup S satisfies (A) for e if *whenever $f < e$, $f \in E$, is such that $eSe \cap J_f$ is completely simple, $faf = f$ for all $a\mathscr{H}e$*, (for in the orthodox case, $\langle E(eSe \cap J_f) \rangle = E(eSe \cap J_f)$).

On the other hand, suppose S is *pseudo-inverse* (sometimes called "locally inverse"), that is, for each $e \in E$, eSe is inverse. Then $eSe \cap J_f$ is completely simple if and only if it is a *group*, equivalently if f is the unique idempotent in J_f which is less than e. In that case $faf = fa$ and of course $\langle E(eSe) \cap J_f \rangle = \{f\}$, so (A) specializes to (A'):

for any e, f in E, $f < e$, such that $eSe \cap J_f$ is a group, $fa = f$ for all $a \mathscr{H} e$.

The hypothesis that $eSe \cap J_f$ be a group has an interesting alternative formulation. Consider the quotient of eSe modulo the ideal $\{x \in eSe: J_x \not\geq J_f\}$ (as in the proof of the last theorem). This inverse semigroup is an ideal extension of the 0-group $H_f \cup \{0\}$ and has identity e. Thus the extension is a *retract* extension [19; Theorem III.4.7] (sometimes called an extension determined by a partial morphism). The requirement that $fa = f$ for all $a \mathscr{H} e$ therefore says that the *retraction* (or partial morphism) *is constant on H_e*.

3. Inverse Semigroups

We now turn to the lattice of θ-classes itself, with regard, in view of Theorem 2.8, to its semimodularity and like properties. In view of Hall's result [9] that for any *semilattice E*, $\Lambda = \Lambda/\theta$ is semimodular, whilst for bands in general this need not be so, we may expect more definitive results for inverse semigroups. Indeed this is the case: we show, for instance, that if S is hypersemisimple and inverse then Λ/θ *is always M-symmetric*, a strong generalization of Hall's theorem. A further reason why the lattices of θ-classes of inverse semigroups are amenable is the following well-known result.

RESULT 3.1. Let S be an inverse semigroup and $\rho, \tau \in \Lambda$. If $e, f \in E$ then $(e, f) \in (\rho \vee \tau)|E$ if and only if there is a sequence

$$e = e_0 \to e_1 \to \cdots \to e_n = f$$

of idempotents of S with each $(e_{i-1}, e_i) \in \rho \cup \tau$.

A general preliminary lemma, whose proof is straightforward is also required.

LEMMA 3.2. *Let S be a regular semigroup and $\rho \in \Lambda$. Then $\Lambda(S/\rho)/\theta \cong [\rho\theta, \omega\theta]$.*

THEOREM 3.3. *Let S be a hypersemisimple inverse semigroup. Then Λ/θ is M-symmetric.*

Proof. Let $\rho, \tau \in \Lambda$ be such that $\rho\theta M \tau\theta$ in Λ/θ. We must show $\tau\theta M \rho\theta$. Now each of these statements is true in Λ/θ if and only if it is true in the interval $[\rho\theta \wedge \tau\theta, \omega\theta]$ of Λ/θ (see Lemma 1.4 of [17]). By Lemma 3.2, this interval is isomorphic with $\Lambda(S/(\rho \cap \tau))/\theta$ and since S is hypersemisimple so is $S/(\rho \cap \tau)$. We may therefore assume, without loss of generality, that $\rho\theta \cap \tau\theta = \iota\theta$ (the zero of Λ/θ).

By assumption, then, for all $\alpha \in \Lambda$ such that $\alpha\theta \subseteq \tau\theta$ we have $((\alpha \vee \rho) \cap \tau)\,\theta = (\alpha\theta \vee \rho\theta) \cap \tau\theta = \alpha\theta$, and we must show, similarly, that for all $\beta \in \Lambda$ such that $\beta\theta \subseteq \rho\theta$ we have $((\beta \vee \tau) \cap \rho)\,\theta = \beta\theta$. In fact it is sufficient to show $((\beta \vee \tau) \cap \rho)|E \subseteq \beta|E$, the reverse inclusion being clear.

So let $(e, f) \in (\beta \vee \tau) \cap \rho$, $e, f \in E$. Without loss of generality $e > f$. By Result 3.1, there is a sequence

$$e = e_0 \to e_1 \to \cdots \to e_n = f$$

of idempotents e_i with each $(e_{i-1}, e_i) \in \beta \cup \tau$. Since $e > f$ we may suppose each $e_i \leqslant e$. Let us assume, as an inductive hypothesis, that any pair of comparable ρ-related idempotents connected by a *shorter* such sequence are β-related, and suppose $(e, f) \notin \beta$. Clearly, then, each $e_i < e$. In that case $(e_{n-1}, f) \notin \beta$, for otherwise $e_{n-1} \rho f$ (since $\beta\theta \subseteq \rho\theta$) and e_{n-1} and e are connected by a shorter sequence, yielding $e \beta e_{n-1} \beta f$. So $e_{n-1} \tau f$ and, similarly, $e \tau e_1$. Now $n \geqslant 2$, for otherwise $e \tau f$ and since $\rho\theta \cap \tau\theta = \iota\theta$, $e = f$, a contradiction.

Let α be the congruence on S generated by those pairs (e_{i-1}, e_i) in τ, $i \geqslant 2$. Then $\alpha \subseteq \tau$, so $\alpha\theta \subseteq \tau\theta$. Moreover, $\alpha \subseteq \rho_L$, where L is the ideal generated by $\{e_1, \ldots, e_n\}$. Since $e \notin L$, $(e, e_1) \notin \alpha$. But $(e, e_1) \in ((\alpha \vee \rho) \cap \tau)|E = \alpha|E$, yielding a contradiction once more. Hence $e \beta f$ and the proof is completed by induction. ∎

Since any semilattice is hypersemisimple and M-symmetry implies semimodularity, the theorem generalizes Hall's theorem [9].

To show that the theorem is false for inverse semigroups in general, let C be a congruence-free inverse semigroup which is not a group, $\phi: C \to D$ an isomorphism and $S = C \cup D$, with linking morphism ϕ. It is easily verified that Λ is then the five-element non-modular lattice, which is not M-symmetric.

Using Lemma 3.2 this also shows that the theorem is false for completely semisimple inverse semigroups, in particular for free inverse semigroups. (From the description given in [3], of $\Lambda(I_1)$, where I_1 is the free *monogenic* inverse semigroup, it is apparent that $\Lambda(I_1)/\theta$ is M-symmetric, however).

Combining Theorems 2.8 and 3.3 with the final comments of Section 2 yields the following.

THEOREM 3.4. *Let S be an inverse semigroup which satisfies* $\min_E$ *or is a semilatice of groups. The following are equivalent.*

(a) *Λ is M-symmetric,*

(b) *Λ is semimodular,*

(c) *S satisfies (A').*

By the remark following Theorem 2.8, each of these is equivalent to double covering in Λ.

The property (A') is very readily tested given meagre knowledge of Green's relations on S. Clearly, for instance, *the congruence lattice of any finite combinatorial inverse semigroup is M-symmetric.* On the other hand a finite E-unitary inverse semigroup S (where $ea = e$, $e^2 = e$, implies $a \in E$) has Λ semimodular if and only if it is an ideal extension of a group by a combinatorial inverse semigroup.

We now investigate the modularity and distributivity of Λ/θ.

THEOREM 3.5. *Let S be a hypersemisimple inverse semigroup. The following are equivalent:*

(a) *Λ/θ is distributive,*

(b) *Λ/θ is modular,*

(c) *for all $e, f, g \in E$ with $e > f$, $e > g$ and f and g incomparable, $(e, f) \in (e, g)^*$.*

Further, if S satisfies $\min_E$ *or is a semilattice of groups each of these is equivalent to*

(B): for any e, f in E, $f < e$, such that $eSe \cap J_f$ is a group, f is comparable with every idempotent of S less than e.

Proof. Clearly (a) implies (b). Now suppose Λ/θ is modular and e, f, g in E are such that $e > f$, $e > g$ and f and g are incomparable. Let I be the ideal SfS, put $\alpha = (e, f)^*$ and $\beta = (e, g)^*$. Since Λ/θ is modular, so is its dual, so that $\alpha\theta M \rho_I \theta$ in the dual of Λ/θ. Thus for all $\gamma \in \Lambda$ such that $\rho_I\theta \subseteq \gamma\theta$,

$$(\gamma \cap (\alpha \vee \rho_I))\,\theta = \gamma\theta \cap (\alpha\theta \vee \rho_I\theta) = (\gamma\theta \cap \alpha\theta) \vee \rho_I\theta = ((\gamma \cap \alpha) \vee \rho_I)\,\theta.$$

In particular, putting $\gamma = \beta \vee \rho_I$,

$$(\beta \vee \rho_I) \cap (\alpha \vee \rho_I)\,\theta\,((\beta \vee \rho_I) \cap \alpha) \vee \rho_I.$$

Now $g = eg\,\alpha fg$ and since f and $fg \in I$,

$$(e, g) \in \beta \cap (\alpha \vee \rho_I) \subseteq (\beta \vee \rho_I) \cap (\alpha \vee \rho_I).$$

Hence $(e, g) \in ((\beta \vee \rho_I) \cap \alpha) \vee \rho_I$. But $e \notin I$, so using Result 1.4 we have $(e, x) \in (\beta \vee \rho_I) \cap \alpha$ for some $x \in I$; and using the same result $e\beta y$ for some $y \in I$, without loss of generality $y \in E$. Now $J_y \leqslant J_f$, so $f \geqslant k$ for some $k \in E \cap J_y$. By Lemma 1.7, $e\beta k$, whence $e\beta f$, that is $(e, f) \in (e, g)^*$.

To prove (c) implies (a), let $\alpha, \beta, \gamma \in \Lambda$. Clearly

$$(\alpha \vee \beta) \cap \gamma \supseteq (\alpha \cap \gamma) \vee (\beta \cap \gamma),$$

so

$$((\alpha \vee \beta) \cap \gamma)\,\theta \geqslant ((\alpha \cap \gamma) \vee (\beta \cap \gamma))\,\theta \quad \text{in } \Lambda/\theta.$$

To prove the reverse inequality, let $(e, f) \in (\alpha \vee \beta) \cap \gamma$, $e, f \in E$, without loss of generality $e > f$. Then there is a sequence

$$e = e_0 \to e_1 \to \cdots \to e_n = f$$

of idempotents e_i of S, where without loss of generality each $e_i \leqslant e$, with each $(e_{i-1}, e_i) \in \alpha \cup \beta$. If $n = 1$ then $(e, f) \in (\alpha \cap \gamma) \cup (\beta \cap \gamma)$. If $n > 1$ suppose that any pair of idempotents belonging to $(\alpha \vee \beta) \cap \gamma$ and connected by a shorter sequence belongs to $(\alpha \cap \gamma) \vee (\beta \cap \gamma)$. If $e_1 \leqslant f$ or if e_1 and f are incomparable then $(e, f) \in (e, e_1)^* \subseteq \alpha \cup \beta$, and so using (c), $(e, f) \in (\alpha \cup \beta) \cap \gamma$. Otherwise $f < e_1 < e$, so $(e, e_1) \in \gamma$. In that case $(e_1, f) \in (\alpha \vee \beta) \cap \gamma$ and e_1 and f are connected by a shorter sequence, so $(e_1, f) \in (\alpha \cap \gamma) \vee (\beta \cap \gamma)$, whence $(e, f) \in (\alpha \cap \gamma) \vee (\beta \cap \gamma)$.

Hence Λ/θ is distributive.

Before proving (c) implies (B) the following preliminary lemma is needed.

LEMMA 3.6. *Let S be an inverse semigroup and $e, f \in E$, $e > f$, such that $eSe \cap J_f$ is a group. If $s \in S$ and $ses^{-1} \geqslant f$ then $sfs^{-1} = f$.*

Proof. If $ses^{-1} \geqslant f$ then $ss^{-1} \geqslant f$, whence $s(s^{-1}fs)\,s^{-1} = f$, so $s^{-1}fs\,\mathscr{D}\,f$. Now $s((s^{-1}fs)\,e)\,s^{-1} = f(ses^{-1}) = f$, so $(s^{-1}fs)\,e = s^{-1}fs$, that is, $s^{-1}fs \in eSe$. Since $eSe \cap J_f$ is a group, $s^{-1}fs = f$ and $f = sfs^{-1}$. ∎

The proof that (c) implies (B) will be valid in any completely semisimple inverse semigroup. Let $e, f \in E$, $e > f$, with $eSe \cap J_f$ a group. Suppose there is an idempotent g of S, $g < e$, which is incomparable with f. By (c), $(e, g) \in (e, f)^*$, so $(f, g) \in (e, f)^*$. There is therefore a sequence

$$f = x_0 \to x_1 \to \cdots \to x_n = g$$

of elements of S with each (x_{i-1}, x_i) of the form $(s_i e t_i, s_i f t_i)$ or $(s_i f t_i, s_i e t_i)$ for some $s_i, t_i \in S^1$. By replacing each x_i by $e_i = x_i x_i^{-1}$ we may assume each (e_{i-1}, e_i) has the form $(s_i e s_i^{-1}, s_i f s_i^{-1})$ or $(s_i f s_i^{-1}, s_i e s_i^{-1})$. Choose such a sequence of minimum length n.

If $(f, e_1) = (s_1 e s_1^{-1}, s_1 f s_1^{-1})$ then by the lemma $e_1 = f$, contradicting minimality. Thus $(f, e_1) = (s_1 f s_1^{-1}, s_1 e s_1^{-1})$, whence $e_1 > f$, (so $e_1 \neq g$). Since S is completely semisimple, $J_{e_1} > J_f$, so $e_1 \neq s_2 f s_2^{-1}$. Thus $(e_1, e_2) = (s_2 e s_2^{-1}, s_2 f s_2^{-1})$. But another application of the lemma then gives $e_2 = f$, again contradicting minimality.

Thus every idempotent g of S, $g < e$, is comparable with f.

Finally suppose S satisfies $\min_E$ or is a semilattice of groups, so that S satisfies (†) (Lemma 2.5). Let $e, f, g \in E$, $e > f$, $e > g$, with f and g incomparable. By (†), $(e, h) \in (e, g)^*$ for some $h \in E$, $h < g$, such that $eSe \cap J_g$ is a group. From (B) it follows that f and h are comparable. Since f and g are incomparable, $f \nleq h$, so $(e>) f > h$ and thus $(e, f) \in (e, g)^*$. Hence S satisfies c). ∎

Combining Theorems 3.5 and 2.8 yields

COROLLARY 3.7. *Let S be an inverse semigroup which satisfies* $\min_E$ *or is a semilattice of groups. Then Λ is modular [distributive] if and only if S satisfies (A') and (B) [and the lattice of normal subgroups of each maximal subgroup is distributive].*

A surprisingly large class of inverse semigroups therefore has modular congruence lattices, for it is easily arranged that (A') and (B) are satisfied vacuously, for example when the semigroup has a zero, no other $\mathscr{J}$-class is a group and whenever $J_e > J_f$, e is greater than every idempotent of J_f.

However $\Lambda(I_1)$ (see earlier) is semimodular [3] although I_1 satisfies (A') and (B) vacuously, so the corollary does not extend to completely semisimple inverse semigroups in general.

In [15] the modularity of congruence lattices of arbitrary *E-unitary* inverse semigroups is studied by means of a different decomposition. The reader is referred there for further details.

When specialized to *semilattices*, the property (B) becomes simply the property that no two incomparable idempotents have a common upper bound, that is, the semilattice is a tree. This result was first obtained by Papert [18] and Dean and Oehmke [2].

4. Decompositions of Λ/θ

The results of this section are somewhat analogous to results (2.1)–(2.3). We begin with a general lemma.

LEMMA 4.1. *Let S be a regular semigroup and $\rho, \tau \in \Lambda$. Then $\rho|E \subseteq \tau|E$ if and only if for all $e, f \in E$,*

(i) *$e\rho f$ and $e > f$ imply $e\tau f$, and*

(ii) *$e\rho f$ and $e\mathscr{R}f$ imply $e\tau f$, and dually.*

Proof. Necessity is clear. Conversely, suppose (i) and (ii) are satisfied, and let $(e, f) \in \rho|E$. Let $h \in S(e, f)$ $(=fV(ef)\,e)$, so $h \in E$, $ef = ehf$ and $h \in fSe \cap E$. Then $h = fhe\,\rho\,ehf = ef\rho e$. Now since $R_h \leqslant R_f$, $hf \in E$, $hf < f$ and $hf\rho f$, so by (i), $hf\tau f$. But $hf\mathscr{R}h$ and $hf\rho h$, so by (ii), $hf\tau h$ and $f\tau h$. Dualizing, we obtain $e\tau h$, so $(e, f) \in \tau|E$. ∎

To specialize this to semigroups satisfying (†) we introduce the following terminology. Given two $\mathscr{R}$-related idempotents e and f in a completely semisimple semigroup we say that e and f are *left similar*, written $L_e \sim L_f$, if for any $x\mathscr{L}e$, the $\mathscr{H}$-classes H_x and H_{xf} (where $xf \in R_x \cap L_f$) are *either both groups or both null*, that is, x is idempotent if and only if xf is.

For each $e \in E$ put $RS(e) = \bigcup \{H_f\colon L_e \sim L_f\}$, a right group contained in R_e, and $RE(e) = E \cap RS(e)$, a right zero semigroup.

The concept of right similarity is defined dually: $LS(e) = \bigcup \{H_f\colon R_e \sim R_f\}$ is a left group contained in L_e, and $LE(e) = L(S) \cap E$ is a left zero semigroup.

Of course if S is completely regular $RS(e) = R_e$ and $LS(e) = L_e$; and if S is inverse, $RS(e) = LS(e) = H_e$.

COROLLARY 4.2. *Let S be a regular semigroup which either satisfies $\min_E$ or is completely regular and let $\rho, \tau \in \Lambda$. Then $\rho|E = \tau|E$ if and only if*

(i) *$\rho|E(eSe) = \tau|E(eSe)$, and*

(ii) *$\rho|RE(e) = \tau|RE(e)$ and $\rho|LE(e) = \tau|LE(e)$, for all $e \in E$.*

Proof. If S is completely regular this is merely a restatement of the previous lemma. On the other hand, suppose S has $\min_E$. Necessity is again clear. Conversely, let $\rho, \tau \in \Lambda$, satisfying (i) and (ii), and suppose $(e, f) \in \rho$. Suppose $e > f$; then $(e, f) \in \rho|E(eSe) = \tau|E(eSe) \subseteq \tau$. If $e\mathscr{R}f$ we may choose $g \in E$ minimally so that $g \leqslant e$ and $g\rho e$. As above, $g\tau e$. Also $gf \in E$ and $gf\mathscr{R}f$, where $gf < f$ and $gf\rho f$, so that $gf\tau f$ similarly.

Suppose H_x is a group but $H_{x(gf)}$ is null, for some $x\mathscr{L}g$. Then $x = xg\rho xgf$, so $x^2\rho(xgf)^2$, where $x^2\mathscr{J}g$ but $J_{(xgf)^2} < J_g$. Then $g\rho y$ for some $y \in S$, $J_y < J_g$, whence by Corollary 1.6 $g\rho h$ for some $h \in E$, $h < g$, contradicting the choice of g. If H_x is null and $H_{x(gf)}$ is a group for some $x\mathscr{L}g$ a similar contradiction is obtained. Hence $gf \in RE(g)$, so by (ii) $g\tau gf$, giving $e\tau f$. The result now follows from duality and the previous lemma. ∎

Under the hypotheses of the corollary there is, therefore, a one-one map

$$\rho\theta \mapsto (\{(\rho|eSe)\,\theta\colon e\in E\},\ \{(\rho|RS(e))\,\theta\colon e\in E\},\ \{(\rho|LS(e))\,\theta\colon e\in E\})$$

of Λ into the direct product of the lattices $\Lambda(eSe)/\theta$, $\Lambda(RS(e))/\theta$ and $\Lambda(LS(e))/\theta$, $e\in E$. Each (complete) morphism $\rho\mapsto\rho|eSe$ of Λ upon $\Lambda(eSe)$ (Proposition 1.3) clearly induces the (complete) morphism $\rho\theta\mapsto(\rho|eSe)\,\theta$ of Λ/θ upon $\Lambda(eSe)/\theta$. We now find necessary and sufficient conditions, like those of Proposition 2.3, in order that the map $\rho\theta\mapsto(\rho|RS(e))\,\theta$ be a morphism. In fact the proof is so similar to that of Proposition 2.3 that we omit it.

PROPOSITION 4.3. *Let S be a completely semisimple semigroup and $e\in E$. The map $\rho\theta\mapsto(\rho|RS(e))\,\theta$ is a morphism if and only if $RE(e)\subseteq e\rho$ for all $\rho\in\Lambda$ such that $e\rho f$ for some $f<e$.*

The proposition therefore yields necessary and sufficient conditions in order that the decomposition above be subdirect-into a subdirect product of the lattice $\Lambda(eSe)/\theta$ and *sublattices* of the lattices $\Lambda(RS(e))/\theta$ and $\Lambda(LS(e))/\theta$, $e\in E$. In general, however, the maps $\rho\theta\mapsto(\rho|RS(e))/\theta$ and their duals will *not* be surjective (see, for example, Example 5.6 of [14]).

Since $RS(e)$ is a right group it is clear that $\Lambda(RS(e))/\theta\cong\Lambda(RE(e))$, under the isomorphism $\rho\theta\mapsto\rho|RE(e)$; and since $RE(e)$ is a right zero semigroup $\Lambda(RE(e))\cong\Pi(RE(e))$, the lattice of partitions (or of equivalences) on $RE(e)$, which is M-symmetric ([6]). However, since *any* lattice can be embedded in the lattice of partitions of some set then unless the corresponding subdirect factor is the whole of $\Lambda(RS(e))/\theta$ no information can in general be obtained.

A further bar to progress in the general case is that we may know little more about the lattices of θ-classes of the subsemigroups eSe then we do about Λ/θ itself.

5. PSEUDO-INVERSE SEMIGROUPS

For pseudo-inverse semigroups each of the above problems may be resolved—each eSe is inverse so that we may apply the results of Section 3, and as we now show, each map $\rho\theta\mapsto(\rho|RS(e))\,\theta$ is surjective. As remarked above, we may replace each $\Lambda(RS(e))/\theta$ by $\Lambda(RE(e))$; we will therefore actually consider from now on the maps $\rho\theta\mapsto\rho|RE(e)$, $e\in E$ (and dually).

The following result, due to T. E. Hall and the author, is crucial. (The example cited at the end of the previous section shows it does not extend to regular semigroups in general.)

RESULT 5.1 [15, Proposition 10.4]. Let S be a pseudo-inverse semigroup and I an ideal of S. Any congruence on I extends to a congruence on S.

LEMMA 5.2. *Let S be a completely semisimple pseudo-inverse semigroup. For each $e \in E$ the maps $\rho\theta \mapsto \rho|RE(e)$ and $\rho\theta \mapsto \rho|LE(e)$ are surjective.*

Proof. Let $e \in E$ and $\pi \in \Lambda(RE(e))$. It is routinely verified, using the complete 0-simplicity of the principal factor associated with J_e, that $\pi^{(1)}$, defined below, is a congruence on SJ_eS extending π: let $(x, y) \in \pi^{(1)}$ if

(i) $x, y \in SJ_eS \backslash J_e$, or

(ii) $x, y \in J_e$ and $x\mathscr{H}y$, or

(iii) $x\mathscr{R}y$, $x \in L_f$, $y \in L_g$, $f\pi g$.

Now $\pi^{(1)}$ extends to a congruence $\pi^{(2)}$ on S, by the previous result, so $\pi^{(2)}|RE(e) = \pi$ and the map $\rho\theta \mapsto \rho|RE(e)$ is surjective. That $\rho\theta \mapsto \rho|LE(e)$ is surjective follows dually. ∎

We now introduce a property similar to (A) (or more precisely (A')). Let $e \in E$. We say S *satisfies* (C) for e *if for any $f \in E, f < e$, such that $eSe \cap J_f$ is a group, $fg = f$ for all $g \in RE(e)$ and $gf = f$ for all $g \in LE(e)$*; S satisfies (C) if it satisfies (C) for each $e \in E$. Note that inverse semigroups satisfy (C) trivially.

The proof of the following theorem, analogous to Theorem 2.7, is so similar to the proof of that theorem that it is also omitted. We first remark that the completely regular pseudo-inverse semigroups are precisely the *normal bands of groups* (equivalently, strong semilattices of completely simple semigroups).

THEOREM 5.3. *Let S be a pseudo-inverse semigroup which either satisfies* $\min_E$ *or is a normal band of groups. The following are equivalent*:

(a) *the map*

$$\rho\theta \mapsto (\{(\rho|eSe)\,\theta\colon e \in E\}, \{\rho|RE(e)\colon e \in E\}, \{\rho|LS(e)\colon e \in E\})$$

is an isomorphism of Λ/θ upon a subdirect product of the lattices $\Lambda(eSe)/\theta$, $\Lambda(RE(e))$ and $\Lambda(LE(e))$, $e \in E$,

(b) *each map $\rho\theta \mapsto \rho|RE(e)$ and $\rho\theta \mapsto \rho|LE(e)$, $e \in E$, is a morphism,*

(c) *S satisfies (C).*

Now given (C), Λ/θ is isomorphic with a subdirect product of M-symmetric lattices (using Theorem 3.3 and the comments at the conclusion of Section 4) and is therefore itself M-symmetric. In direct analogy with the result of Section 1, we now show that semimodularity implies (C).

THEOREM 5.4. *Let S be a pseudo-inverse semigroup. If S is hypersemisimple and Λ/θ is semimodular then S satisfies (C). Hence if, further, S has* $\min_E$ *or is a normal band of groups then M-symmetry and semimodularity of Λ/θ are equivalent, and each is equivalent to (C).*

Proof. Only the first statement remains to be proved. So suppose Λ/θ is semimodular and let $e, f \in E$, $f < e$, with $eSe \cap J_f$ a group. By Result 5.1, $\Lambda(SJ_eS)$ is isomorphic with an ideal of Λ (under the monomorphism $\rho \mapsto \rho \cup \iota_s$), and so $\Lambda(SJ_eS)/\theta$ is isomorphic with an ideal of Λ/θ and is therefore also semimodular. By also factoring out the ideal $\{x \in S\colon J_x \ngeqslant J_f\}$ we may assume, without loss of generality, that J_e is the greatest $\mathscr{J}$-class of S and J_f is the least non-zero $\mathscr{J}$-class of S.

Suppose $e\rho f$ for some $\rho \in \Lambda$. By Theorem 5.3 and Proposition 4.3 it is sufficient to show $RE(e) \subseteq e\rho$ (the dual result following similarly). Denote by I the ideal $S \backslash J_e$, and let J be a $\mathscr{J}$-class of S such that $J_e > J \geqslant J_f$. By [8, Theorem 1], $e > h$ for some $h \in E(J)$, and $h \geqslant k$ for some $h \in E(J_f)$. Since f is the *unique* idempotent in J_f less than e, ($eSe \cap J_f$ being a group), $e > h \geqslant f$ and so $(h, k) \in \rho$. Now observe that if $g \in RE(e)$ then $fg\mathscr{R}f$ and $g\rho fg$. Thus if $x\rho 0$ for some $x\mathscr{J}f$, so that $J_f \subseteq 0\rho$, $e\rho g$, as required. So from now on we assume that $\rho | (J_f \cup \{0\})$ is 0-restricted. Let $T = S/(\rho \cap \rho_I) = J_{\bar{e}} \cup J_{\bar{f}} \cup \{0\}$, where we put $\bar{x} = x(\rho \cap \rho_I)$ for all $x \in S$. Again, $\Lambda(T)$ is semimodular.

Clearly $\bar{e} > \bar{f}$. Moreover if $\bar{e} > \bar{h}\mathscr{J}\bar{f}$, $\bar{h} \in E(T)$, then $\bar{h} = \bar{e}\bar{h}\bar{e}$, that is, $h\rho ehe$. Thus $h\rho fhf$. By assumption $fhf \neq 0$, so $fhf\mathscr{H}f$ and $h\rho f$, that is $\bar{h} = \bar{f}$. Hence $\bar{e}$ is again greater than a unique idempotent of $J_{\bar{f}}$, and the same is therefore true for each idempotent of $J_{\bar{e}}$.

Put $L = \bar{I}$ $(= J_{\bar{f}} \cup \{0\})$. By [19, Theorem III.4.7], T is a *retract* extension of L, determined by a retraction ψ of T upon L; if $\bar{h}$ is a nonzero idempotent of T then $\bar{h}\psi$ is the unique nonzero idempotent of L less than or equal to $\bar{h}$, that is, $\bar{h}\psi = \bar{l}$, where l is the unique idempotent in J_f less than or equal to h. Denote by η the congruence $\psi\psi^{-1}$ on T. Clearly $\eta \cap \rho_L = \iota$, and since $\bar{e}\psi = \bar{f} = \bar{f}\psi$, $(\bar{e}, \bar{f})^* \subseteq \eta$. Now if $\bar{h} \in E(J_{\bar{e}})$ then $(h, k)^* \in (\bar{e}, \bar{f})^*$ for some $k \in E(L)$, so $(\bar{h}, \bar{h}\psi) \in (\bar{e}, \bar{f})^*$. Thus for any $x\mathscr{J}\bar{e}$, with inverse x', $(x, (xx')\psi \cdot x) \in (e, f)^*$, that is, $(x, x\psi) \in (\bar{e}, \bar{f})^*$. Hence $\eta = (\bar{e}, \bar{f})^*$. By Zorn's lemma there is a congruence γ on T, maximal such that $\gamma \subset \eta$ and $(\bar{e}, \bar{f}) \notin \gamma$. Thus $\eta \succ \gamma$ in $\Lambda(T)$ and, since $\gamma\theta \neq \eta\theta$, $\eta\theta \succ \gamma\theta$ in $\Lambda(T)/\theta$.

Note that $\gamma \subseteq \mathscr{J}_T$. Thus, using Result 1.4, $\gamma \vee \rho_L = \gamma \cup \rho_L$, so that $(\gamma \vee \rho_L) \cup \eta = \gamma$. By semimodularity of $\Lambda(T)/\theta$, $(\gamma \vee \rho_L)\theta \prec ((\gamma \vee \rho_L) \vee \eta)\theta = \omega\theta$.

Let $g \in RE(e)$. Since $\rho \cap \rho_I$ is trivial on J_e, $\bar{g} \in RE(\bar{e})$. Let π be the congruence on the right zero semigroup $RE(\bar{e})$ which identifies $\bar{e}$ and $\bar{g}$. Define $\pi^{(1)}$ as in the proof of Lemma 5.2: since $T = TJ_{\bar{e}}T$, $\pi^{(1)}$ is a congruence on T extending π and containing ρ_L. Further, $\pi^{(1)} \subseteq \mathscr{J}$ and so $\gamma \vee \pi^{(1)} \subseteq \mathscr{J}$. Since, therefore, $(\gamma \vee \rho_L)\theta \subseteq (\gamma \vee \pi^{(1)})\theta \subset \omega\theta$, we obtain

$(\gamma \vee \rho_L)\theta = (\gamma \vee \pi^{(1)})\theta$ and in particular $(\bar{e}, \bar{g}) \in \gamma \vee \rho_L$. But $\gamma \vee \rho_L = \gamma \cup \rho_L$, so $\bar{e}\gamma\bar{g}$ and thus $\bar{e}\eta\bar{g}$, that is, $\bar{e}\psi = \bar{g}\psi$. But $\bar{e}\psi = \bar{f}$ and $\bar{g}\psi = \overline{fg}$, so $(f, fg) \in \rho$. Since $e\rho f$ and $g\rho fg$, we have $e\rho g$, completing the proof. ∎

We remark that with rather more difficulty it may be shown that double covering in Λ/θ may replace semimodularity in the statement of the theorem.

Combining this theorem with Theorem 2.8 and the final comments of Section 2 yields the following generalization of Theorem 3.4.

COROLLARY 5.5. *Let S be a pseudo-inverse semigroup which either satisfies* $\min_E$ *or is a normal band of groups. The following are equivalent*:

(a) *Λ is M-symmetric,*

(b) *Λ is semimodular,*

(c) *S satisfies (A') and (C).*

The conjunction of (A') and (C) is clearly the following: *for any $e, f \in E$, $e > f$, such that $eSe \cap J_f$ is a group, $fx = f$ for all $x \in RS(e)$ and $sf = f$ for all $x \in LS(e)$.* Thus for normal bands of groups (in which $eSe \cap J_f$ is *always* a group), M-symmetry and semimodularity (and double covering) of Λ are equivalent to the constancy of every non-identical linking morphism. So Corollary 5.5 extends Theorem 3.5 of [4].

Finally we consider the modularity and distributivity of Λ/θ and, therefore, of Λ (using Theorem 2.8). If S satisfies (C) then Λ/θ is isomorphic with a subdirect product of the lattices $\Lambda(eSe)/\theta$, $\Lambda(RE(e))$ and $\Lambda(LE(e))$, $e \in E$. Since eSe is inverse, modularity and distributivity of $\Lambda(eSe)/\theta$ are equivalent, and equivalent to (B) on eSe (Theorem 3.5). Moreover S itself satisfies (B) if and only if each eSe satisfies (B). Further, $RE(e)$ being a right zero semigroup, with $\Lambda(RE(e)) \cong \Pi(RE(e))$ (see Section 4), $\Lambda(RE(e))$ is modular [distributive] if and only if $|RE(e)| \leqslant 3$ $[|RE(e)| \leqslant 2]$ ([6, Section IV.4].)

This yields the following generalization of Corollary 3.7.

COROLLARY 5.6. *Let S be a pseudo-inverse semigroup which either satisfies* $\min_E$ *or is a normal band of groups. Then Λ is modular [distributive] if and only if S satisfies (A'), (B) and (C) and S contains at most* 3 *[at most* 2*] right similar or left similar idempotents [and the lattice of normal subgroups of each maximal subgroup is distributive].*

When specialized to normal bands of groups, again, (B) amounts to requiring that the structure semilattice be a tree (using the final comments of Section 3). Thus we obtain, for instance, the characterization of normal bands with distributive congruence lattice found in [5, Theorem 6].

6. θ-Modular Bands of Groups

In this section we show the results of Spitznagel in [23] can be deduced from those in Section 2.

A *band of groups* is a completely regular semigroup on which $\mathscr{H}$ is a congruence. A regular semigroup is called *θ-modular* [22] if the conditions $\rho \subseteq \tau$, $\rho\theta\tau$, $\rho \vee \pi = \tau \vee \pi$ and $\rho \cap \pi = \tau \cap \pi$, where $\rho, \tau, \pi \in \Lambda$, together imply $\rho = \tau$.

The equivalence of (i) and (ii) in the next theorem is the content of Theorems 3.14 and 3.15 of [22]. (For a derivation from a general theory of "Φ-modularity" see Theorem 7.4 of [15]). We will prove the remaining equivalences, thus deducing the main results of [23]: Proposition 3.1 and Theorem 4.9 (the equivalences (i)$\Leftrightarrow$(iii) and (i)$\Leftrightarrow$(iv) respectively). In Theorem 4.9 of [23] an alternative form of (A) derived at the end of Section 2 is used.

THEOREM 6.1. *Let S be a band of groups. The following are equivalent:*

(i) *S is θ-modular,*

(ii) *the map $\rho \mapsto ((\rho \vee \mathscr{H})/\mathscr{H}, \rho \cap \mathscr{H})$ is an isomorphism (of Λ upon a subdirect product of $\Lambda(S/\mathscr{H})$ and $[\iota, \mathscr{H}]$),*

(iii) *$H_e \subseteq e\rho$ for all $\rho \in \Lambda$ such that $e\rho f$ for some $f < e$,*

(iv) *S satisfies (A),*

(v) *each map $\rho \mapsto \rho | H_e$, $e \in E$, is a morphism.*

Proof. The equivalence of (iv) and (v) is immediate from Theorem 2.7 and that of (iii) and (iv) from Proposition 2.3.

Suppose S satisfies (ii). Then the map $\rho \mapsto \rho \cap H$ is a morphism of Λ upon $[\iota, \mathscr{H}]$. From Theorem 2.1 of [14] we deduce that in any completely regular semigroup each map $\tau \mapsto \tau | H_e$, $e \in E$, is a morphism of $[\iota, \mathscr{H}]$ into $\Lambda(H_e)$, and so S satisfies (v).

To show (v) implies (i), let $\rho, \tau, \pi \in \Lambda$, such that $\rho \subseteq \tau$, $\rho\theta\tau$, $\rho \vee \pi = \tau \vee \pi$ and $\rho \cap \pi = \tau \cap \pi$. Let $e \in E$. Then by (v) we have $\rho | H_e \cap \pi | H_e = \tau | H_e \cap \pi | H_e$, $\rho | H_e \vee \pi | H_e = \tau | H_e \vee \pi | H_e$ and $\rho | H_e \subseteq \tau | H_e$, whence by modularity of $\Lambda(H_e)$ we obtain $\rho | H_e = \tau | H_e$. That $\rho = \tau$ now follows from Corollary 2.2. ∎

The special cases Theorem 4.10 (for orthodox bands of groups) and Theorem 4.14 (for normal bands of groups) in [23] now are immediate from the concluding remarks of Section 2.

Addendum

As an application of the foregoing, we derive the characterization of finite inverse "perfect" semigroups with modular congruence lattice obtained by H. Hamilton and T. Tamura, as part of a paper (*J. Austral. Math. Soc. Ser. A* **32** (1982), 114–128) which appeared after the submission of the present paper.

A semigroup S is *perfect* if for each congruence ρ on S the product of two ρ classes is again a ρ class. In Theorem 1.11 of the cited paper the authors essentially proved that a finite inverse semigroup is perfect if and only if it is either (I) a chain of groups, with surjective structure mappings, or (II) an ideal extension of a Brandt semigroup by such a chain of groups, in such a way that each idempotent of the chain of groups is above every idempotent of the Brandt semigroup.

It is clear, then, that for idempotents $e > f$ of such a semigroup, $eSe \cap J_f$ can be a group only when J_f is itself a group. Thus (see Theorem 3.5) (B) is always satisfied and Λ/θ is distributive; applying Theorem 2.8, it follows that semimodularity, M symmetry and modularity of Λ are all equivalent to (A), that is, to (A') (see the final paragraph of Section 2). From the remarks above, and the surjectivity of the structure mappings, it is immediate that (A') is equivalent to the property that at most one group $\mathscr{H}$ class is nontrivial: this is Theorem 4.8 of the cited paper.

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